JOURNAL OF APPROXIMATION THEORY **89**, 96–113 (1997)
ARTICLE NO. AT963030

Strong Uniqueness in Restricted Rational Approximation

Chong Li

Department of Mathematics, Hangzhou Institute of Commerce, Hangzhou, People's Republic of China

and

G. A. Watson

Department of Mathematics and Computer Science, University of Dundee, Dundee DD14HN, Scotland

Communicated by Günther Nurnberger

Received August 9, 1995; accepted February 27, 1996

The problem is considered of approximating continuous functions in the uniform norm by rational functions whose denominators are bounded from above and below. A general theory of strong uniqueness is presented.

1. INTRODUCTION

Let X be a compact Hausdorff space and $C(X)$ the space of real continuous functions on X. For $f \in C(X)$, define

$$\|f\| = \max_{x \in X} \{|f(x)| : x \in X\}.$$

Let $\{\phi_1, ..., \phi_n\}$ and $\{\psi_1, ..., \psi_m\}$ be two linearly independent subsets of $C(X)$, and define

$$\mathbf{P} = \text{span}\{\phi_1, ..., \phi_n\}$$
$$\mathbf{Q} = \text{span}\{\psi_1, ..., \psi_m\}$$
$$\mathbf{R} = \{p/q : p \in \mathbf{P}, q \in \mathbf{Q}, q(x) > 0 : \forall x \in X\},$$
$$\mathbf{R}_{\mu, \nu} = \{p/q \in \mathbf{R} : \mu(x) \leqslant q(x) \leqslant \nu(x) : \forall x \in X\},$$

where μ, ν are two given elements of $C(X)$ with $0 < \mu(x) < \nu(x)$ for all $x \in X$. We will assume that $\mathbf{R}_{\mu, \nu}$ is non-empty.

0021-9045/97 $25.00

DEFINITION 1. For $f \in C(X)$, $r_f = p_f/q_f \in \mathbf{R}_{\mu,\nu}$ is called a best approximation to f from $\mathbf{R}_{\mu,\nu}$ if

$$\|f - r_f\| = d(f, \mathbf{R}_{\mu,\nu}) = \inf_{r \in \mathbf{R}_{\mu,\nu}} \|f - r\|.$$

The set of all best approximations to f from $\mathbf{R}_{\mu,\nu}$ is denoted by $P_{\mathbf{R}_{\mu,\nu}}(f)$.

DEFINITION 2. If $r_f \in \mathbf{R}_{\mu,\nu}$, and if there exists $c > 0$ such that for any $r \in \mathbf{R}_{\mu,\nu}$

$$\|f - r\| \geqslant \|f - r_f\| + c\,\|r - r_f\|, \tag{1.1}$$

then r_f is said to be a strongly unique best approximation to f from $\mathbf{R}_{\mu,\nu}$ or equivalently r_f is strongly unique.

The problem of approximating $f \in C(X)$ from $\mathbf{R}_{\mu,\nu}$ was apparently first studied by Dunham [3]. The study was motivated by the desire to improve some unsatisfactory features of approximation from $\mathbf{R}$, normally associated with denominators going to zero at points in X. It is not sufficient simply to provide a lower bound on q, because of the possibility of multiplying both numerator and denominator by an arbitrary constant. Topological properties were established in [3], and some characterization results (of Kolmogorov type) were given. The more difficult question of uniqueness was also addressed, and in particular it was shown that uniqueness is only possible if $m \leqslant 2$. Other studies concerned with characterization and uniqueness of constrained rational approximation have involved different approximating sets (for example [4, 5]), and further results concerning, for example, strong uniqueness for $\mathbf{R}_{\mu,\nu}$ have not, so far, been available. The intention of this paper is to present a general theory of strong uniqueness for best approximation to f from $\mathbf{R}_{\mu,\nu}$. In particular, several sufficient conditions are given such that the best approximation is strongly unique: some questions raised in [3] are therefore answered. The work can also be interpreted as generalising results of Nurnberger [7].

2. STRONG UNIQUENESS

It is necessary to introduce some more notation. For $f \in C(X)$, $r_f = p_f/q_f \in \mathbf{R}_{\mu,\nu}$, let

$$X_0 = \{x \in X: |f(x) - r_f(x)| = \|f - r_f\|\},$$

$$X_\mu = \{x \in X: q_f(x) = \mu(x)\},$$

$$X_\nu = \{x \in X: q_f(x) = \nu(x)\},$$

and let

$$\sigma(x) = \operatorname{sign}(f - r_f)(x) = \begin{cases} 0, & (f - r_f)(x) = 0 \\ \dfrac{(f - r_f)(x)}{|(f - r_f)(x)|}, & (f - r_f)(x) \neq 0. \end{cases}$$

We will now define a uniqueness element of a set and give a known characterization result which is required later.

Definition 3 [6]. For a subset $G \subset C(X)$, $g \in G$ is called a uniqueness element of G if for any $f \in C(X)$, $g \in P_G(f)$ implies that g is a unique best approximation to f from G.

Lemma 1 [6, Theorem 3]. *The following statements are equivalent.*

(1) $r^* \in \mathbf{R}_{\mu, \nu}$ *is a uniqueness element of* $\mathbf{R}_{\mu, \nu}$.

(2) *For any* $f \in C(X) \backslash \mathbf{R}_{\mu, \nu}$,

$$r^* \in P_{\mathbf{R}_{\mu, \nu}}(f) \qquad \text{if and only if} \quad \max_{x \in X_0} \sigma(x)(r^* - r)(x) > 0 \quad \forall r \in \mathbf{R}_{\mu, \nu} \backslash \{r^*\}.$$

Theorem 1. *Let* $f \in C(X) \backslash \mathbf{R}_{\mu, \nu}$, *and* $r_f \in \mathbf{R}_{\mu, \nu}$. *Then,* r_f *is a strongly unique best approximation to f from* $\mathbf{R}_{\mu, \nu}$ *if and only if there exists* $c > 0$ *such that for any* $r \in \mathbf{R}_{\mu, \nu}$

$$\max_{x \in X_0} \sigma(x)(r_f - r)(x) \geqslant c \, \|r_f - r\|. \tag{2.1}$$

Proof. Since for any $r \in \mathbf{R}_{\mu, \nu}$

$$\|f - r\| - \|f - r_f\| \geqslant \max_{x \in X_0} \sigma(x)(r_f - r)(x), \tag{2.3}$$

then (2.1) is clearly sufficient.

Now assume that r_f is a strongly unique best approximation to f from $\mathbf{R}_{\mu, \nu}$. Let $r = p/q \in \mathbf{R}_{\mu, \nu}$ be arbitrary, $r \neq r_f$. For $t > 0$, define

$$r_t = \frac{(1 - t)\, p_f + tp}{(1 - t)\, q_f + tq}.$$

For any $t > 0$, let $x_t \in X$ be such that

$$\|f - r_t\| = |f(x_t) - r_t(x_t)|. \tag{2.2}$$

Since $\|r_t - r_f\| \to 0$ as $t \to 0+$, without loss of generality we may assume that $x_t \to x_0 \in X_0$, and for $t > 0$ sufficiently small

$$\operatorname{sign}(f - r_t)(x_t) = \sigma(x_0).$$

Now for $t > 0$ sufficiently small

$$\begin{aligned}\|f - r_t\| - \|f - r_f\| &\leqslant |(f - r_t)(x_t)| - |(f - r_f)(x_t)| \\ &= \sigma(x_0)(r_f - r_t)(x_t) \\ &= \sigma(x_0)\, t \frac{q(x_t)}{(1-t)\, q_f(x_t) + tq(x_t)} (r_f - r)(x_t).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\|f - r_t\| - \|f - r_f\|}{t} &\leqslant \sigma(x_0) \frac{q(x_t)}{(1-t)\, q_f(x_t) + tq(x_t)} (r_f - r)(x_t) \\ &\leqslant \max_{x \in X_0} \sigma(x) \frac{q(x_t)}{(1-t)\, q_f(x_t) + tq(x_t)} (r_f - r)(x_t). \qquad (2.3)\end{aligned}$$

Also, for any $x \in X_0$,

$$\begin{aligned}\|f - r_t\| - \|f - r_f\| &\geqslant |(f - r_t)(x)| - |(f - r_f)(x)| \\ &\geqslant \sigma(x)(f - r_t)(x) - \sigma(x)(f - r_f)(x) \\ &= \sigma(x)(r_f - r_t)(x) \\ &= \sigma(x)\, t \frac{q(x)}{(1-t)\, q_f(x) + tq(x)} (r_f - r)(x).\end{aligned}$$

Therefore, for all $x \in X_0$,

$$\frac{\|f - r_t\| - \|f - r_f\|}{t} \geqslant \sigma(x) \frac{q(x)}{(1-t)\, q_f(x) + tq(x)} (r_f - r)(x). \qquad (2.4)$$

Let $t \to 0+$. Then from (2.3) and (2.4), it follows that

$$\lim_{t \to 0+} \frac{\|f - r_t\| - \|f - r_f\|}{t} = \max_{x \in X_0} \sigma(x) \frac{q(x)}{q_f(x)} (r_f - r)(x). \qquad (2.5)$$

Define

$$\tau(f, r_f, r) = \lim_{t \to 0+} \frac{\|f - r_t\| - \|f - r_f\|}{t}. \qquad (2.6)$$

Then

$$\tau(f, r_f, r) \geqslant \lim_{t \to 0+} \frac{c\,\|r_t - r_f\|}{t},$$

using (1.1),

$$= c\left\|\frac{pq_f - p_f q}{q_f^2}\right\|$$

$$\geqslant c \min_{x \in X} \left|\frac{q(x)}{q_f(x)}\right| \|r - r_f\|$$

$$\geqslant c \min_{x \in X} \frac{\mu(x)}{\nu(x)} \|r - r_f\|.$$

Therefore, there exists $c' > 0$ such that for any $r \in \mathbf{R}_{\mu, \nu}$

$$\tau(f, r_f, r) \geqslant c'\,\|r - r_f\|. \tag{2.7}$$

Since $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$,

$$\max_{x \in X_0} \sigma(x)(r_f - r)(x) \geqslant 0 \qquad \forall r \in \mathbf{R}_{\mu, \nu}.$$

Further, for all $r \in \mathbf{R}_{\mu, \nu}$,

$$\max_{x \in X_0} \sigma(x)(r_f - r)(x) \geqslant \min_{x \in X_0} \frac{q_f(x)}{q(x)} \max_{x \in X_0} \frac{q(x)}{q_f(x)} \sigma(x)(r_f - r)(x)$$

$$= \min_{x \in X_0} \frac{q_f(x)}{q(x)} \tau(f, r_f, r)$$

$$\geqslant c' \min_{x \in X_0} \frac{\mu(x)}{\nu(x)} \|r - r_f\|, \tag{2.8}$$

using (2.5), (2.6) and (2.7). This establishes (2.1) and the proof is complete. ∎

We now introduce some further notation which is needed for what follows. Firstly, for any $x \in X$, define

$$\hat{g}(x) = (\phi_1(x), ..., \phi_n(x), r_f(x)\,\psi_1(x), ..., r_f(x)\,\psi_m(x)) \in R^{m+n},$$

$$\hat{h}(x) = (0, ..., 0, \psi_1(x), ..., \psi_m(x)) \in R^{m+n}.$$

Then, we can define the sets

$$S_1 = \{\sigma(x)\ \hat{g}(x) : x \in X_0\},$$
$$S_2 = \{-\hat{h}(x) : x \in X_\mu\},$$
$$S_3 = \{\hat{h}(x) : x \in X_\nu\},$$

with

$$S = S_1 \cup S_2 \cup S_3.$$

For any $r_f \in \mathbf{R}_{\mu,\nu}$, let

$$G_{r_f} = \mathbf{P} + r_f\,\mathbf{Q} = \{p + r_f\,q : p \in \mathbf{P},\ q \in \mathbf{Q}\},$$
$$G^*_{r_f} = \{p + r_f\,q \in G_{r_f} : q(x) \geqslant 0,\ \forall x \in X_\mu;\ q(x) \leqslant 0,\ \forall x \in X_\nu\}.$$

DEFINITION 4. $\mathbf{R}_{\mu,\nu}$ satisfies the *Interior Condition* if there exists $q_0 \in Q$ such that $\mu(x) < q_0(x) < \nu(x)$ for all $x \in X$.

Remark. This condition is effectively a constraint qualification analogous to the Slater constraint qualification in nonlinear programming.

THEOREM 2. *Suppose that*

$$\max_{x \in X_0} \sigma(x)\ g(x) > 0 \qquad \forall g \in \overline{G^*_{r_f}} \backslash \{0\}, \tag{2.9}$$

where the bar denotes closure of the set. Then $r_f = p_f/g_f$ is a strongly unique best approximation to f from $\mathbf{R}_{\mu,\nu}$. If the Interior Condition holds, then (2.9) is also a necessary condition for r_f to be a strongly unique best approximation.

Proof. Suppose that (2.9) holds for some $r_f \in \mathbf{R}_{\mu,\nu}$. Since $\overline{G^*_{r_f}}$ is a finite dimensional closed convex cone, the set

$$\left\{\frac{g}{\|g\|} : g \in \overline{G^*_{r_f}},\ g \neq 0\right\}$$

is a compact subset of $\overline{G^*_{r_f}}$. It follows from (2.9) that there exists $c > 0$ such that

$$\max_{x \in X_0} \sigma(x)\ g(x) \geqslant c\ \|g\| \qquad \forall g \in \overline{G^*_{r_f}}. \tag{2.10}$$

Since for any $r = p/q \in \mathbf{R}_{\mu,\nu}$

$$\sigma(x)(r_f - r)(x) = \frac{1}{q(x)}\,\sigma(x)(p_f - p + r_f(q - q_f))(x),$$

consequently

$$q(x)\,\sigma(x)(r_f-r)(x)=\sigma(x)(p_f-p+r_f(q-q_f))(x).$$

Thus using (2.10),

$$\begin{aligned}\max_{x\in X_0} q(x)\,\sigma(x)(r_f-r)(x)&=\max_{x\in X_0}\sigma(x)(p_f-p+r_f(q-q_f))(x)\\&\geqslant c\,\|p_f-p+r_f(q-q_f)\|\\&\geqslant c\min_{x\in X_0} q(x)\,\|r_f-r\|.\end{aligned}$$

Further

$$\begin{aligned}\max_{x\in X_0}\sigma(x)(r_f-r)(x)&\geqslant\min_{x\in X_0}\frac{1}{q(x)}\max_{x\in X_0}\sigma(x)\,q(x)(r_f-r)(x)\\&\geqslant c\min_{x\in X_0} q(x)\min_{x\in X_0}\frac{1}{q(x)}\|r_f-r\|\\&\geqslant c\min_{x\in X_0}\mu(x)\min_{x\in X_0}\frac{1}{v(x)}\|r_f-r\|.\end{aligned}$$

It follows from Theorem 1 that r_f is a strongly unique best approximation to f from $\mathbf{R}_{\mu,v}$.

Now let the Interior Condition hold. Also, assume that $r_f\in\mathbf{R}_{\mu,v}$ is a strongly unique best approximation, but there exists $g\in\overline{G^*_{r_f}}$, $g\neq 0$ such that

$$\max_{x\in X_0}\sigma(x)\frac{g(x)}{\|g\|}=0.$$

Let $g_n=p_n+r_f\,q_n\in G^*_{r_f}$, with $\|g_n-g\|\to 0$ as $n\to\infty$, and let

$$g_n^\lambda=p_n+r_f(q_n+\lambda(q_0-q_f)),$$

where $q_0\in Q$ with $\mu(x)<q_0(x)<v(x)$ for any $x\in X$, which exists by assumption. Since $g_n^\lambda\to g$ uniformly as $\lambda\to 0+$ and $n\to\infty$, then for any $\varepsilon>0$ there exists $\lambda_\varepsilon>0$ and integer $N_\varepsilon>0$ such that for any $0<\lambda\leqslant\lambda_\varepsilon$, $n\geqslant N_\varepsilon$,

$$\max_{x\in X_0}\sigma(x)\frac{g_n^\lambda(x)}{\|g_n^\lambda\|}\leqslant\varepsilon. \tag{2.11}$$

Let

$$q_n^\lambda = q_n + \lambda(q_0 - q_f), \qquad \lambda > 0,$$

$$r_n^t = \frac{p_f - tp_n}{q_f + tq_n^\lambda}, \qquad t > 0.$$

Then, for any $\lambda > 0$, and any n, there exists $t_n^\lambda > 0$ such that for any $0 < t \leqslant t_n^\lambda$, $r_n^t \in \mathbf{R}_{\mu,\nu}$. In fact, since X_μ is compact and $q_n^\lambda(x) > 0$ for any $x \in X_\mu$, there exists an open subset $W_\mu \subset X$, with $X_\mu \subset W_\mu$ such that

$$q_n^\lambda(x) > 0 \qquad \text{for any} \quad x \in W_\mu.$$

Thus, $K_\mu = X \backslash W_\mu$ is compact and $K_\mu \cap X_\mu = \phi$. Obviously, for any $x \in W_\mu$, $t > 0$,

$$(q_f + tq_n^\lambda)(x) \geqslant q_f(x) \geqslant \mu(x).$$

Now define

$$\alpha = \min\{q_f(x) - \mu(x) : x \in K_\mu\}.$$

Then $\alpha > 0$. Let

$$\hat{t}_n^\lambda = \alpha / \max\{1, \|q_n^\lambda\|\}.$$

Then for any $0 < t \leqslant \hat{t}_n^\lambda$, $x \in K_\mu$,

$$\begin{aligned} q_f(x) + tq_n^\lambda(x) &\geqslant q_f(x) - t\,|q_n^\lambda(x)| \\ &\geqslant q_f(x) - \alpha\,|q_n^\lambda(x)|/\max\{1, \|q_n^\lambda\|\} \\ &\geqslant q_f(x) - \alpha \\ &\geqslant \mu(x). \end{aligned}$$

Thus, when $0 < t \leqslant \hat{t}_n^\lambda$,

$$(q_f + tq_n^\lambda)(x) \geqslant \mu(x) \qquad \forall x \in X.$$

By a similar argument, for any $\lambda > 0$, and n, there exists $\tilde{t}_n^\lambda > 0$ such that for $0 < t \leqslant \tilde{t}_n^\lambda$

$$(q_f + tq_n^\lambda)(x) \leqslant \nu(x) \qquad \forall x \in X.$$

Thus, we have proved that for any $\lambda > 0$ and n, there exists $t_n^\lambda > 0$ such that for any $0 < t \leqslant t_n^\lambda$, $r_n^t \in \mathbf{R}_{\mu,\nu}$.

Now, for any $x \in X_0$, $0 < \lambda < \lambda_\varepsilon$, $n > N_\varepsilon$, and $0 < t \leqslant t_n^\lambda$,

$$\frac{\sigma(x)(r_f - r_n^t)(x)}{\|r_f - r_n^t\|} = \frac{\sigma(x)(r_f\, q_n^\lambda + p)(x)}{(q_f + tq_n^\lambda)(x) \left\| \dfrac{r_f\, q_n^\lambda + p}{q_f + tq_n^\lambda} \right\|}$$

$$\leqslant \varepsilon \frac{\|r_f\, q_n^\lambda + p\|}{(q_f + tq_n^\lambda)(x)} \Big/ \left\| \frac{r_f\, q_n^\lambda + p}{q_f + tq_n^\lambda} \right\| \qquad \text{since} \quad g_n^\lambda = r_f\, q_n^\lambda + p_n,$$

$$\leqslant \varepsilon \left\{ \frac{1}{\mu(x)} \right\} \Big/ \min \left| \frac{1}{(q_f + tq_n^\lambda)(x)} \right|$$

$$\leqslant \varepsilon \max_{x \in X} \nu(x) / \min_{x \in X} \mu(x).$$

Let $\varepsilon \to 0+$. Then this gives a contradiction using Theorem 1 and the proof is complete. ∎

Remark. The usefulness of this result is obviously enhanced if $G_{r_f}^*$ is closed. This will be a consequence of a unique representation of an element of $G_{r_f}^*$, in other words if $g = p + r_f q \in G_{r_f}^*$ with $g = 0$ implies that $p = 0$ and $q = 0$.

Corollary 1. *If*

$$0 \in \operatorname{IntCo}(S),$$

where $\operatorname{IntCo}(S)$ *denotes the interior of the convex hull of the set* S, *then* r_f *is a strongly unique best approximation to* f *from* $\mathbf{R}_{\mu,\nu}$.

Proof. Let $0 \in \operatorname{IntCo}(S)$. Let $g = p + r_f q \in G_{r_f}^*$ be such that

$$\max_{x \in X_0} \sigma(x)\, g(x) \leqslant 0. \tag{2.12}$$

Let

$$p = a_1 \phi_1 + \cdots + a_n \phi_n,$$
$$q = b_1 \psi_1 + \cdots + b_m \psi_m,$$

and let

$$z = (a_1, ..., a_n, b_1, ..., b_m)^T \in R^{m+n}.$$

Then (2.12) together with the definition of $G_{r_f}^*$ imply that

$$\langle z, s \rangle \leqslant 0 \qquad \forall s \in S,$$

using a standard inner product notation. Since $0 \in \operatorname{IntCo}(S)$, for $\delta > 0$ sufficiently small, it follows that

$$\delta z \in \operatorname{Co}(S).$$

Using Caratheodory's Theorem, for some $k \leqslant m+n+1$, there exist $s_1, ..., s_k \in S$, $\lambda_1 \geqslant 0, ..., \lambda_k \geqslant 0$ with $\sum_{i=1}^{k} \lambda_i = 1$ such that

$$\delta z = \sum_{i=1}^{k} \lambda_i s_i.$$

Thus

$$0 \leqslant \delta \langle z, z \rangle = \langle z, \delta z \rangle = \sum_{i=1}^{k} \lambda_i \langle z, s_i \rangle \leqslant 0.$$

This implies that $z = 0$. Thus if $g = p + r_f\, q \in G^*_{r_f}$, with $g = 0$, it follows that $p = 0$ and $q = 0$. Thus from the above Remark, $G^*_{r_f}$ is closed. Thus (2.9) is satisfied, and the result follows from Theorem 2. ∎

For subsequent results, we require the following characterization theorem.

THEOREM 3. *Suppose that the Interior Condition holds. For any* $f \in C(X) \backslash \mathbf{R}_{\mu, \nu}$, $r_f \in P_{\mathbf{R}_{\mu, \nu}}(f)$ *if and only if there exist* $x_1, ..., x_{k_1} \in X_0$, $\alpha_1 > 0, ..., \alpha_{k_1} > 0$, $y_1, ..., y_l \in X_\mu$, $y_{l+1} \cdots y_{k_2} \in X_\nu$, $\beta_1 > 0, ..., \beta_{k_2} > 0$ *with* $k_1 + k_2 \leqslant m+n+1$, *such that*

$$\sum_{i=1}^{k_1} \alpha_i \sigma(x_i)\, p(x_i) = 0 \qquad \forall p \in \mathbf{P}, \tag{2.13}$$

$$\sum_{i=1}^{k_1} \alpha_i \sigma(x_i)\, r_f(x_i)\, q(x_i) = \sum_{i=1}^{l} \beta_i q(y_i) - \sum_{i=l+1}^{k_2} \beta_i q(y_i) \qquad \forall q \in \mathbf{Q}. \tag{2.14}$$

Proof. Clearly, there exist $x_1, ..., x_{k_1} \in X_0$, $y_1, ..., y_l \in X_\mu$, $y_{l+1}, ..., y_{k_2} \in X_\nu$, $\alpha_1 > 0, ..., \alpha_{k_1} > 0$, $\beta_1 > 0, ..., \beta_{k_2} > 0$ such that (2.13) and (2.14) hold if and only if

$$0 \in \operatorname{Co}(S). \tag{2.15}$$

Using a standard separation result (for example [2], p. 19), (2.15) is equivalent to the non-existence of any $p \in \mathbf{P}$, $q \in \mathbf{Q}$ such that

$$(1) \quad \sigma(x)(p + r_f q)(x) < 0 \qquad \forall x \in X_0, \tag{2.16}$$

$$(2) \quad q(x) > 0 \quad \forall x \in X_\mu; \quad q(x) < 0, \quad \forall x \in X_\nu. \tag{2.17}$$

Thus, it is sufficient to prove that $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$ if and only if there is no $p \in \mathbf{P}$, $q \in \mathbf{Q}$ such that (2.16) and (2.17) hold.

First, suppose that there exist $p \in \mathbf{P}$, $q \in \mathbf{Q}$ such that (2.16) and (2.17) hold. Then arguing as in the proof of Theorem 2, for $\lambda > 0$ sufficiently small, $r_\lambda = (p_f - \lambda p)/(q_f + \lambda q) \in \mathbf{R}_{\mu,\nu}$. But

$$\sigma(x)(r_f - r_\lambda)(x) = \lambda \frac{\sigma(x)(p + r_f q)}{q_f + \lambda q} < 0 \qquad \forall x \in X_0$$

which implies (see [3]) that $r_f \notin P_{\mathbf{R}_{\mu,\nu}}(f)$. Thus (2.13) and (2.14) are necessary.

Next suppose that (2.13) and (2.14) hold, so that (2.16) and (2.17) do not hold for any $p \in \mathbf{P}$, $q \in \mathbf{Q}$. Then if $r_f \notin P_{\mathbf{R}_{\mu,\nu}}(f)$ it follows from [3] that there exists $r_1 = p_1/q_1 \in \mathbf{R}_{\nu,\mu}$ such that

$$\sigma(x)(r_f - r_1)(x) < 0 \qquad \forall x \in X_0. \tag{2.18}$$

Let

$$p = p_f - p_1, \qquad q = q_1 - q_f + \eta(q_0 - q_f)$$

where

$$0 < \eta < \inf_{x \in X_0} |r_1(x) - r_f(x)| \Big/ \left\| \frac{r_f(q_f - q_0)}{q_1} \right\| \qquad \text{and} \qquad q_0 \in Q$$

with $\mu(x) < q_0(x) < \nu(x)$ for all $x \in X$. Then, $p \in \mathbf{P}$, $q \in \mathbf{Q}$ satisfy (2.16) and (2.17). This is a contradiction which establishes the sufficiency of (2.13) and (2.14) and completes the proof. ∎

COROLLARY 2. *Suppose that the Interior Condition holds. If $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$ and 0 is a uniqueness element of $\overline{G^*_{r_f}}$ then r_f is strongly unique.*

Proof. Let the Interior Condition hold, let $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$ and let 0 be a uniqueness element of $\overline{G^*_{r_f}}$. Then by Theorem 3 there exists $x_1, ..., x_{k_1} \in X_0$, $y_1, ..., y_l \in X_\mu$, $y_{l+1}, ..., y_{k_2} \in X_\nu$, $\alpha_1 > 0, ..., \alpha_{k_1} > 0$, $\beta_1 > 0, ..., \beta_{k_2} > 0$ such that (2.13) and (2.14) hold. Thus, for any $g = p + r_f q \in \overline{G^*_{r_f}}$, since $q(x) \geqslant 0$, for all $x \in X_\mu$, $q(x) \leqslant 0$, for all $x \in X_\nu$, we have

$$\sum_{i=1}^{k_1} \alpha_i \sigma(x_i)\, g(x_i) = \sum_{i=1}^{l} \beta_i q(y_i) - \sum_{i=l+1}^{k_2} \beta_i q(y_i) \geqslant 0.$$

Hence

$$\max_{x \in X_0} \sigma(x)\, g(x) \geqslant 0 \qquad \forall g \in \overline{G^*_{r_f}}.$$

It follows from the convexity of $\overline{G^*_{r_f}}$ that 0 is a best approximation to $f - r_f$ from $\overline{G^*_{r_f}}$. The result follows from Lemma 1 and Theorem 2. ∎

Lemma 2. *Assume that the Interior Condition holds. Let* $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$, *so that by Theorem* 3 (2.13) *and* (2.14) *are satisfied. Let* $g = p + r_f q \in G^*_{r_f}$ *with*

$$\max_{x \in X_0} \sigma(x)\, g(x) \leqslant 0. \tag{2.19}$$

Then

$$g(x_i) = 0, \qquad i = 1, 2, ..., k_1,$$

$$q(y_i) = 0, \qquad i = 1, 2, ..., k_2.$$

Proof. Using Theorem 3, (2.19) implies that

$$\begin{aligned} 0 &\geqslant \sum_{i=1}^{k_1} \alpha_i \sigma(x_i)(p + r_f\, q)(x_i) \\ &= \sum_{i=1}^{k_1} \alpha_i \sigma(x_i)\, p(x_i) + \sum_{i=1}^{k_1} \alpha_i \sigma(x_i)\, r_f(x_i)\, q(x_i) \\ &= \sum_{i=1}^{l} \beta_i q(y_i) - \sum_{i=l+1}^{k_2} \beta_i q(y_i). \end{aligned}$$

Since $q(y_i) \geqslant 0$, $i = 1, 2, ..., l$, $q(y_i) \leqslant 0$, $i = l+1, ..., k_2$, it follows that

$$q(y_i) = 0 \qquad i = 1, 2, ..., k_2.$$

It is an immediate consequence of this that

$$g(x_i) = 0, \qquad i = 1, 2, ..., k_1,$$

and the result is proved. ∎

This result is useful in a number of ways. In particular, it enables us to give some conditions under which the set $G^*_{r_f}$ is closed.

Theorem 4. *Let the Interior Condition hold, and let* $P = \Pi_{n-1}$, $Q = \Pi_{m-1}$, *where* Π_k *is the space of polynomials with degree* $\leqslant k$. *For given* f, *let* $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$, $r_f \notin P_{\mathbf{R}}(f)$, *with* r_f *irreducible. Then if* $\min\{m - \partial q_f, n - \partial p_f\} \leqslant 1$ (*where* ∂ *is used to denote the actual degree of the polynomial*), $G^*_{r_f}$ *is closed.*

Proof. Let the stated assumptions hold, and let $g=p+r_f\,q\in G^*_{r_f}$ with $g=0$. Obviously (2.19) holds, so that in particular, by the proof of Lemma 2,

$$\sum_{i=1}^{l} \beta_i q(y_i) - \sum_{i=l+1}^{k_2} \beta_i q(y_i) = 0, \qquad i=1, 2, ..., k_2, \tag{2.20}$$

where k_2 and $Y_0=\{y_1, ..., y_{k_2}\}$ are given by Theorem 3. Now since $p+qp_f/q_f=0$, any zeros of q are also zeros of p, and the zeros of p must include those of p_f. Thus there must exist a polynomial c such that

$$p=cp_f, \qquad q=-cq_f$$

where

$$\partial c \leqslant \min\{m-\partial q_f, n-\partial p_f\} \leqslant 1.$$

If $k_2 \leqslant 1$, then it follows from Theorem 3 that we must have $r_f\in P_{\mathbf{R}_0}(f)$ where

$$\mathbf{R}_0=\{r\in\mathbf{R}: \mu(x)\leqslant q(x): \forall x\in X\} \quad \text{or} \quad \mathbf{R}_0=\{r\in\mathbf{R}: q(x)\leqslant \nu(x): \forall x\in X\}.$$

Therefore $r_f\in P_{\mathbf{R}}(f)$ on appropriate scaling, which is a contradiction. Thus $Y_0\cap X_\mu\neq\phi$ and $Y_0\cap X_\nu\neq\phi$, $k_2\geqslant 2$, and since (2.20) implies that $c(y_i)=0$, $i=1, 2, ..., k_2$, it follows that $c=0$, so that $p=q=0$. Thus $G^*_{r_f}$ is closed. ∎

THEOREM 5. *Let the Interior Condition hold, and P and Q be Haar subspaces. For given f, let $r_f\in P_{\mathbf{R}_{\mu,\nu}}(f)$, $r_f\notin P_{\mathbf{R}}(f)$. Then if* $\min\{m, n\}\leqslant 2$, *$G^*_{r_f}$ is closed.*

Proof. Let the stated assumptions hold, and let $g=p+r_f\,q\in G^*_{r_f}$ with $g=0$. As in the proof of Theorem 4, Y_0 contains at least 2 points. But $q(y_i)=0$ implies that $p(y_i)=0$, $i=1, 2, ..., k_2$, and so $p=q=0$ by the condition on the dimensions. The result follows. ∎

We now present some further conditions which lead to strong uniqueness.

THEOREM 6. *Assume that the Interior Condition holds. Let $r_f\in P_{\mathbf{R}_{\mu,\nu}}(f)$, and let $Y_0=\{y_1, ..., y_{k_2}\}\subset X_\mu\cup X_\nu$, β_i, $i=1, ..., k_2$ be given by Theorem 3. Write*

$$G^0_{r_f}=\left\{p+r_f\,q\in G_{r_f}: \sum_{i=1}^{l} \beta_i q(y_i) - \sum_{i=l+1}^{k_2} \beta_i q(y_i)=0\right\}.$$

If 0 is a uniqueness element of $G^0_{r_f}$, then r_f is strongly unique.

Proof. Let $r_f \in P_{\mathbf{R}_{\mu,\nu}}$. Assuming that the interior condition holds, let $g = p + r_f q \in \overline{G^*_{r_f}}$, with

$$\max_{x \in X_0} \sigma(x)\, g(x) \leqslant 0. \tag{2.21}$$

Then, by Lemma 2,

$$q(y_i) = 0 \qquad i = 1, 2, ..., k_2,$$

and so $g \in G^0_{r_f}$. It follows that if 0 is a uniqueness element of $G^0_{r_f}$, then (2.9) must hold. By Theorem 2, r_f is strongly unique. ∎

THEOREM 7. *Assume that the Interior Condition holds. Let $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$, and let $Y_0 = \{y_1, ..., y_{k_2}\} \subset X_\mu \cup X_\nu$, β_i, $i = 1, ..., k_2$ be given by Theorem 3. Write*

$$\tilde{G}_{r_f} = \{p + r_f q \in G_{r_f} : q(y_i) = 0 : i = 1, 2, ..., k_2\}. \tag{2.22}$$

(a) *If $G^*_{r_f}$ is closed and 0 is a uniqueness element of $\tilde{G}_{r_f}$, then r_f is strongly unique.*

(b) *If P is a Haar subspace and also $\tilde{G}_{r_f}$ is a Haar subspace of dimension $\leqslant n + 1$, then r_f is strongly unique.*

Proof. The proof of (a) is similar to that of Theorem 6. Consider (b). Let $g = p + r_f\, q \in \overline{G^*_{r_f}}$ such that (2.19) holds. Lemma 2 shows that $g \in \tilde{G}_{r_f}$. Then using Theorem 3, since P is Haar, (2.13) implies that $k_1 \geqslant n + 1$. It follows from Lemma 2 that $g = 0$. Thus (2.9) must hold, and by Theorem 2, r_f is strongly unique. ∎

COROLLARY 3. *Assume that the Interior Condition holds, and let P be a Haar subspace. Let $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$, and let G_{r_f} be a Haar subspace of $C(X)$. If one of the following two conditions holds then r_f is strongly unique.*

(1) $d(f, \mathbf{R}_{\mu,\nu}) = d(f, \mathbf{R})$,

(2) *either X_μ or X_ν is empty.*

Proof. Since (1) is implied by (2), we only need to prove the result in the case that $d(f, \mathbf{R}_{\mu,\nu}) = d(f, \mathbf{R})$. Suppose, therefore, that this is true.

Take μ', $\nu' \in C(X)$ with $0 < \mu'(x) < \mu(x) < \nu(x) < \nu'(x)$ for all $x \in X$. Then $r_f \in P_{\mathbf{R}_{\mu',\nu'}}(f)$ and $X_{\mu'} = X_{\nu'} = \phi$. Hence, with μ, ν replaced by μ', ν', we have

$$\tilde{G}_{r_f} = G_{r_f},$$

and so $\tilde{G}_{r_f}$ is a Haar subspace of $C(X)$. Thus $0 \in \tilde{G}_{r_f}$ is a uniqueness element of $\tilde{G}_{r_f}$. The result follows from Theorem 7. ∎

Note that Corollary 3 is not true for non-restricted rational approximation. For example if **P** and **Q** are spaces of polynomials, then the best approximation is strongly unique if and only if it is normal (Barrar and Loeb [1]).

COROLLARY 4. *Suppose that* **P** *and* **Q** *are both Haar subspaces of* $C(X)$. *Let* $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$. *If* $m \leqslant 2$, *then* r_f *is strongly unique.*

Proof. Note that the Interior Condition holds since **Q** is a Haar subspace of dimension 2. It follows that Theorem 3 can be used. If condition (1) of Corollary 3 holds, the result is immediate, so assume that $d(f, \mathbf{R}_{\mu,\nu}) > d(f, \mathbf{R})$. Let $Y_0 = \{y_1, ..., y_{k_2}\}$ be given by Theorem 3. Then as in the proof of Theorem 4, $Y_0 \cap X_\mu \neq \phi$ and $Y_0 \cap X_\nu \neq \phi$ which implies that Y_0 contains at least two points. Thus, $\tilde{G}_{r_f} = \mathbf{P}$, so $\tilde{G}_{r_f}$ is a Haar subspace of $C(X)$ of dimension n. The result then follows from Theorem 7. ∎

Now let $X = [a, b]$, $\mathbf{P} = \Pi_{n-1}$, $\mathbf{Q} = \Pi_{m-1}$. Then we have the following result, which gives an affirmative answer to a question raised in [3].

COROLLARY 5. *Let* $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$. *If any one of the following three conditions holds, then* r_f *is strongly unique.*

(1) $d(f, \mathbf{R}_{\mu,\nu}) = d(f, \mathbf{R})$,

(2) $X_\mu = \phi$ *or* $X_\nu = \phi$,

(3) $m \leqslant 2$.

THEOREM 8. *Let* $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$. *Let any of the following conditions hold:*

(1) 0 *is a unique best approximation to* $f - r_f$ *from* $\overline{G^*_{rf}}$.

(2) *The Interior Condition holds and* 0 *is a unique best approximation to* $f - r_f$ *from* G^0_{rf}.

(3) *The Interior Condition holds,* 0 *is a unique best approximation to* $f - r_f$ *from* $\tilde{G}_{rf}$, *and* G^*_{rf} *is closed.*

Then there exists $\{f_n\} \subset C(X)$ *such that* $\|f_n - f\| \to 0$ *and* r_f *is a strongly unique best approximation to* f_n *from* $\mathbf{R}_{\mu,\nu}$.

Proof. This is similar to the proof of Theorem 2.1 of Smarzewski [8]. ∎

It is an open question whether or not $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$ is a unique approximation to f implies that there exists $\{f_n\} \subset C(X)$ such that $\|f_n - f\| \to 0$ and r_f is a strongly unique best approximation to f_n.

The question also arises: is it possible to characterize uniqueness elements of $R_{\mu,\nu}$ in terms of strong uniqueness? We conclude with two examples which show that these properties are not implied by each other. The following lemma is useful.

LEMMA 3. *Let* $X=[a,b]$, $P=\Pi_{n-1}$, $Q=\Pi_2$, $\mu, \nu \in C^1[a,b]$. *Then if* $r_f=p_f/q_f \in \mathbf{R}_{\mu,\nu}$ *with* $X_\mu \subset (a,b)$ *or* $X_\mu \subset (a,b)$, *then* r_f *is a uniqueness element of* $\mathbf{R}_{\mu,\nu}$.

Proof. Let $r_f=p_f/q_f$, $r_1=p_1/q_1 \in P_{\mathbf{R}_{\mu,\nu}}(f)$. Then

$$r_0=\frac{1/2(p_1+p_f)}{1/2(q_1+q_f)} \in P_{\mathbf{R}_{\mu,\nu}}(f).$$

Assume that $1/2(q_1+q_f)(x)=\mu(x)$ for some $x \in X$ and $1/2(q_1+q_f)(y)=\nu(y)$ for some $y \in X$. Then $q_1(x)=q_f(x)$, $q_1(y)=q_f(y)$, and $x \in X_\mu$, $y \in X_\nu$. By the assumptions,

$$q_1'(x)=q_f'(x) \qquad \text{or} \qquad q_1'(y)=q_f'(y).$$

Thus $q_1=q_f$.

Now since $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f)$, we must have

$$\max_{x \in X_0} \sigma(x) \frac{p_f(x)-p(x)}{q_f} \geqslant 0, \qquad \text{for all} \quad p \in \Pi_n.$$

It follows that

$$\max_{x \in X_0} \sigma(x)(p_f(x)-p(x))>0 \qquad \text{for all} \quad p \in \Pi_n \backslash \{p_f\},$$

and so $p_1=p_f$. The result is proved. ∎

EXAMPLE 1. Let $X=[-1/2,1]$, $P=\Pi_0$, $Q=\Pi_2$, $\mu=1$, $\nu=2$. Let $f \in C[-1/2,1]$ be given by

$$f(x)=\begin{cases} 4x+2 & -1/2 \leqslant x \leqslant 0 \\ -1/2(5x-4) & 0 \leqslant x \leqslant 1 \end{cases}.$$

Let

$$r_f=\frac{1}{1+x^2}.$$

Then $\|f-r_f\|=1$, and

$$X_0=\{0,1\}, \qquad X_\mu=\{0\}, \qquad X_\nu=\{1\}, \qquad \sigma(0)=1, \qquad \sigma(1)=-1.$$

Let $\alpha_1=\alpha_2=1$, $\beta_1=1$, $\beta_2=1/2$. Then

$$\alpha_1\sigma(0)\,p(0)+\alpha_2\sigma(1)\,p(1)=0, \qquad \text{for all} \quad p\in\Pi_0,$$

$$\alpha_1\sigma(0)\,r_f(0)\,q(0)+\alpha_2\sigma(1)\,r_f(1)\,q(1)=\beta_1 q(0)-\beta_2 q(1), \qquad \text{for all} \quad q\in\Pi_2.$$

Thus by Theorem 3, $r_f\in P_{\mathbf{R}_{\mu,\nu}}(f)$, and by Lemma 3, r_f is a uniqueness element of $\mathbf{R}_{\mu,\nu}$.

Now let

$$g=(x-x^2)\,r_f.$$

Then $g\in G^*_{r_f}$, and further

$$\sigma(0)\,g(0)=0, \qquad \sigma(1)\,g(1)=0.$$

Thus by Theorem 2, r_f is not strongly unique.

This example shows that strong uniqueness is not necessarily implied by the existence of a uniqueness element. The next example shows that the reverse implication is also false.

Example 2. Let $X=[-1,1]$, $P=\Pi_0$, $Q=\Pi_2$,

$$\mu(x)=\begin{cases}1/2x^2+1/2x+1 & -1\leqslant x\leqslant 0\\ 1 & 0\leqslant x\leqslant 1\end{cases}, \qquad \nu=2.$$

Let

$$f_1(x)=\begin{cases}1/2(5x+4) & -1\leqslant x\leqslant 0\\ -1/2(5x-4) & 0\leqslant x\leqslant 1\end{cases}, \qquad r_f(x)=\frac{1}{1+x^2}.$$

Then $\|f_1-r_f\|=1$, with $X_0=\{-1,0,1\}$, $X_\mu=\{0\}$, $X_\nu=\{-1,1\}$, $\sigma(-1)=-1$, $\sigma(0)=1$, $\sigma(1)=-1$. It is readily seen that $r_f\in P_{\mathbf{R}_{\mu,\nu}}(f_1)$. Further, since $G^*_{r_f}$ is closed, (2.19) is easily verified, and so r_f is strongly unique.

Now let

$$r^*_f=\frac{1}{1/2x^2+1/2x+1},$$

and define $f_2(x)$ by

$$f_2=\begin{cases}r^*_f & -1\leqslant x\leqslant -1/2\\ 2+12x/7 & -1/2\leqslant x\leqslant 0.\\ 2-5x/2 & 0\leqslant x\leqslant 1\end{cases}$$

Further, for r_f^*, f_2: $X_0 = \{0, 1\}$, $X_\mu = [-1, 0]$, $X_\nu = \{1\}$. It is readily verified that $r_f^* \in P_{\mathbf{R}_{\mu,\nu}}(f_2)$, with $\|r_f^* - f_2\| = 1$. (In Theorem 3, take $l = 1$, with $y_1 = 0$.) In addition, $\|r_f - f_2\| = 1$, so that $r_f \in P_{\mathbf{R}_{\mu,\nu}}(f_2)$. It follows that r_f is not a uniqueness element of $\mathbf{R}_{\mu,\nu}$.

3. CONCLUSIONS

We have given various conditions which lead to strong uniqueness of the constrained rational Chebyshev approximation problem. The relationship between a uniqueness element and strong uniqueness has been investigated, and it is shown by examples that these are not equivalent. This is not really surprising because the property of being a uniqueness element is a global property (it holds for all f), while strong uniqueness is a point property (valid only for a fixed f).

ACKNOWLEDGMENTS

This work was partially supported by the National Science Foundation of China and the China State Major Project for Basic Researchers.

REFERENCES

1. R. B. Barrar and H. L. Loeb, On the continuity of the nonlinear Tschebyscheff operator, *Pacific J. Math.* **32** (1970), 593–601.
2. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
3. C. B. Dunham, Chebyshev approximation by rationals with constrained denominators, *J. Approx. Theory* **37** (1983), 5–11.
4. C. B. Dunham, Chebyshev approximation by restricted rationals, *Approx. Theory and Appl.* **1**, No. 2 (1985), 111–118.
5. E. H. Kaufman, Jr. and G. D. Taylor, Uniform approximation by rational functions having restricted denominators, *J. Approx. Theory* **32** (1981), 9–26.
6. C. Li and G. A. Watson, Characterization of a best and a unique best approximation from constrained rationals, *Comput. Math. Appl.* **30** (1995), 51–57.
7. G. Nurnberger, Unicity and strong unicity in approximation theory, *J. Approx. Theory* **26** (1979), 54–70.
8. R. Smarzewski, Strong unicity of best approximations in $L_\infty(S, \Sigma, \mu)$, *Proc. Amer. Math. Soc.* **103**, No. 1 (1988), 113–116.